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2007 J. Phys. A: Math. Theor. 40 5083

(http://iopscience.iop.org/1751-8121/40/19/010)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 5083–5103 doi:10.1088/1751-8113/40/19/010

Group classification of the general second-order evolution equation: semi-simple invariance groups

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Received 2 January 2007, in final form 26 March 2007
Published 24 April 2007
Online at stacks.iop.org/JPhysA/40/5083

Abstract

In this paper, we consider the problem of group classification of the generic second-order evolution equation in one spatial variable. We construct all inequivalent evolution equations whose invariance groups are either semi-simple or semi-direct products of semi-simple and solvable Lie groups. The obtained lists of invariant equations contain both already known equations and the broad classes of new evolution equations possessing nontrivial Lie symmetry.

PACS numbers: 02.20.Qs, 02.20.Sv, 02.20.Jr
Mathematics Subject Classification: 35K55, 70G65, 76M60

1. Introduction

Utilization of group properties of partial differential equations (PDEs) has already become a universal and convenient tool for analysis of these equations. Clearly for this technique to work, equations under study should actually have nontrivial group properties. Within this viewpoint, the whole class of differential equations splits into two subclasses of equations with nontrivial symmetry, $\mathcal{S}$, and without any symmetries. The whole history of exploration of symmetries of differential equations is the sequence of attempts to extend the class $\mathcal{S}$ by modifying somehow the classical concept of Lie symmetry.

As the paper title implies, we restrict our analysis to equations from $\mathcal{S}$. Moreover, we narrow the meaning of admitted symmetry by considering invariance with respect to Lie transformation groups only. The basic facts and all the necessary information about group analysis of differential equations can be found in [1–4].

In this paper, we study the general evolution equation

$$u_t = F(t, x, u, u_x, u_{xx}) \tag{1.1}$$

in order to get an answer to the following seemingly simple question: is it possible to describe all possible functions F such that equation (1.1) admits nontrivial Lie transformation group? By nontrivial symmetry group we mean a Lie transformation group which is at least one parameter. Hereafter, $u = u(t, x)$, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and F is an arbitrary sufficiently smooth function.

The first paper on group classification of a subclass of linear equations from the class (1.1) was published by the creator of the theory of transformation groups Sophus Lie as early as in 1881 [5]. However, the real boom of interest in group classification of differential equations was initiated by the Ovsyannikov's paper [6]. It was followed by numerous publications (see [9–20] and the references therein) analysing various specific subclasses of the general class of evolution equations (1.1). A detailed account of group properties of the equations considered in the above papers can be found in [8, 21, 22].

Surprisingly, there is still no ultimate solution of the classification problem for the general evolution equation (1.1). The main reason is that the class of equations (1.1) is too general for the traditional Ovsyannikov's classification method to be practical. This method is not very efficient when the class of equations under study involves functions of several variables.

Recently, we developed an efficient approach to solving group classification problem for low-dimensional partial differential equations. It enabled us to classify the broad classes of heat conductivity [21, 23], Schrödinger [27], third-order evolution [28] and wave [29] equations admitting nontrivial Lie symmetry. Note that some elements of this approach were utilized earlier by Fushchych and Serov [24], Gagnon and Winternitz [25] and Zhdanov *et al* [26] in order to perform symmetry classification of the nonlinear d'Alembert, Schrödinger and multi-component wave equations, correspondingly.

In the present paper, we apply the method of [21] to obtain exhaustive classification of evolution equations that admit n-parameter Lie transformation group for all possible values of $n \geqslant 1$. As group classification of linear PDEs of the form (1.1) has already been performed in [5], we consider essentially nonlinear evolution equations only. By 'essentially nonlinear' we mean PDEs (1.1) that cannot be linearized by point transformations of the space of the variables t, x, u.

2. Preliminary group analysis of equation (1.1)

It is a common knowledge that the most general (in Lie's sense) transformation group admitted by (1.1) is generated by the infinitesimal operators

$$v = \tau\partial_t + \xi\partial_x + \eta\partial_u. \tag{2.1}$$

Here $\tau = \tau(t, x, u)$, $\xi = \xi(t, x, u)$, $\eta = \eta(t, x, u)$ are arbitrary smooth functions defined in the space of $V = R^2 \times R^1$ of two independent $R^2 = \langle t, x\rangle$ and one dependent $R^1 = \langle u\rangle$ variables.

Constructing the second prolongation of the infinitesimal operator v we get

$$\widetilde{v} = v + \varphi^t \frac{\partial}{\partial u_t} + \varphi^x \frac{\partial}{\partial u_x} + \varphi^{tt} \frac{\partial}{\partial u_{tt}} + \varphi^{tx} \frac{\partial}{\partial u_{tx}} + \varphi^{xx} \frac{\partial}{\partial u_{xx}},$$

where

$$\begin{aligned}
\varphi^t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\varphi^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\varphi^{xx} &= D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\\
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,
\end{aligned}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots .$$

We do not give the formulae for the coefficients φ^{tt}, φ^{tx} since they are not used in the following.

Acting by $\widetilde{v}$ on equation (1.1) we arrive at the following invariance criterion:

$$\left(\varphi^t - \tau F_t - \xi F_x - \eta F_u - \varphi^x F_{u_x} - \varphi^{xx} F_{u_{xx}}\right)\Big|_{u_t \to F(t,x,u,u_x,u_{xx})} = 0. \qquad (2.2)$$

The subscript formula in (2.2) means that one needs to replace u_{tx} with $D_x F$, u_{tt} with $D_t F$ and u_t with F in the expression within the parentheses.

If we construct the general solution of (2.2), then we obtain the most general (local) Lie transformation group admitted by equation (1.1). Note that this group is also called the classical Lie symmetry of nonlinear PDE (1.1).

Analysing relations (2.2) we prove the following technical assertion.

Assertion 2.1. *The most general invariance group of equation (1.1) is generated by the infinitesimal operators*

$$v = \tau(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \qquad (2.3)$$

the functions τ, ξ, η *and* F *satisfying the following equation:*

$$\begin{aligned} \eta_t - u_x\xi_t + (\eta_u - \tau_t - u_x\xi_u)F = {} & \left[\eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_u\right]F_{u_x} \\ & + \left[\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_x^2(\eta_{uu} - 2\xi_{xu}) - u_x^3\xi_{uu}\right. \\ & \left. + u_{xx}(\eta_u - 2\xi_x) - 3u_x u_{xx}\xi_u\right]F_{u_{xx}} + \tau F_t + \xi F_x + \eta F_u. \end{aligned} \qquad (2.4)$$

As the forms of unknown functions τ, ξ, η depend essentially on the function F, it is customary to call (2.4) the classifying equation.

Now the problem of group classification of equation (1.1) becomes entirely algorithmic. It reduces to constructing all possible solutions of a single partial differential equation (2.4). It is that simple. The difficulty, however, is that we have to deal with the under-determined system of partial differential equations that consists of one equation for three unknown functions τ, ξ, η. To make things even more complicated, this equation contains unknown function F of variables t, x, u, u_x, u_{xx}, which is to be determined as well.

To proceed any further we need extra information, either on the form of the function F or on the form of the functions τ, ξ, η which would narrow the class of invariant equations or the set of possible symmetries. Say, we may put $F = f(u)u_{xx} + \frac{\mathrm{d}f(u)}{\mathrm{d}u}u_x^2$, split the resulting equation by the variables u_x, u_{xx} and obtain an over-determined system of partial differential equations for τ, ξ, η. Solving the latter yields the classical Ovsyannikov's classification result [6].

An alternative approach would be fixing *a priori* the symmetry group of equation (1.1) and solving (2.4) for this specific choice of coefficients of the infinitesimal group operator, τ, ξ, η. To this end, one can also use the popular nowadays technique called conventionally 'the method of moving frames'. The latter can be used efficiently, if one succeeds in obtaining the explicit form of the finite group transformations of the infinitesimal symmetry group whose differential invariants are to be constructed (see, e.g., [7] and the references therein).

The problem with the above approaches is that some functions F providing for extension of symmetry group of equation (1.1) might be lost. To prevent this from happening, the classifying equation should be the only source of constraints on the form of unknown functions τ, ξ, η, F. Another difficulty is that the method of moving frames is inefficient when the group under consideration is infinite parameter and this is exactly the case we are dealing with.

In [26] we suggested an elegant way to approach this problem based on the fact that the general solution (τ, ξ, η) of PDE (2.4) can always be represented as a linear combination of its basis solutions, $v_a = (\tau_a, \xi_a, \eta_a)$, $a = 1, \ldots, n$, forming a Lie algebra ℓ_n. Consequently, if we succeed in describing all possible subalgebras of the infinite-dimensional Lie algebra ℓ_∞ generated by operators (2.3) and solve for each of them the classifying equation (2.4), then the problem of group classification of initial equation (1.1) will be completely solved.

So that we can reformulate the problem of group classification of equation (1.1) in a purely algebraic way. Namely, to solve the classification problem we need to

- construct all subalgebras of the infinite-dimensional Lie algebra ℓ_∞ and
- select those subalgebras whose basis elements satisfy the classifying equation (2.4).

Saying it another way, we can replace (2.2) with the (possibly infinite) set of systems of PDEs:

$$\begin{cases} \text{Equation (2.4)}|_{\mathbf{v}\to\mathbf{v}_a}, & a = 1, \ldots, n, \\ Q_i\tau_j - Q_j\tau_i = \sum_{k=1}^{n} C_{ij}^k \tau_k, \\ Q_i\xi_j - Q_j\xi_i = \sum_{k=1}^{n} C_{ij}^k \xi_k, \\ Q_i\eta_j - Q_j\eta_i = \sum_{k=1}^{n} C_{ij}^k \eta_k, \end{cases} \tag{2.5}$$

where $Q_i = \tau_i\partial_t + \xi_i\partial_x + \eta_i\partial_u$, C_{ij}^k are structure constants of a Lie algebra ℓ_n, $i, j = 1, \ldots, n$ and $n = 1, 2, 3, \ldots$.

If we solve over-determined system of PDEs (2.5) for *all* possible dimensions $n \geqslant 1$ of *all* admissible Lie algebras ℓ_n, then the problem of group classification of equation (1.1) is completely solved. Consequently, group classification of the general evolution equation (1.1) reduces to integrating over-determined systems of PDEs (2.5) for all $n = 1, 2, \ldots, n_{\max}$, where $n_{\max}$ is the maximal dimension of the Lie algebra admitted by the equation under study. Note that $n_{\max}$ may be equal to ∞.

According to the Magadeev theorem [30], we have either $n_{\max} \leqslant 7$ or $n_{\max} = \infty$. And what is more, in the latter case the corresponding invariant equation is mapped to a linear PDE by a contact transformation. So to describe all essentially nonlinear evolution equations (i.e., those inequivalent to linear ones) one needs, in fact, to consider all possible Lie algebras of the dimension up to 7. However, combining our Lie algebraic classification approach with the Ovsyannikov method it suffices to consider Lie algebras of the dimension $n < 5$ only. At this point, the classical Mubarakzyanov results [37, 39] come into play. He described all inequivalent abstract Lie algebras of the dimension $n < 6$. This means that the structure constants C_{ij}^k in (2.5) are already known.

Let us mention here the Reid's procedure of calculating the Lie algebra admitted by PDE without integrating determining equations [39]. The basic idea of his approach is investigating compatibility of systems (2.5) thus deriving the admissible forms of the structure constants C_{ij}^k. In our mind a more natural approach is actually to integrate equations (2.5) so that the compatibility conditions come as a by-product.

Summarizing we conclude that if we

(1) construct all realizations of all subalgebras of ℓ_∞ by operators, coefficients of which satisfy equation (2.5), and
(2) prove that these are maximal invariance algebras of (1.1),

then the problem of group classification of equation (1.1) is completely solved.

Thus it is clear *what* to do to achieve complete group classification of the generic class of PDEs (1.1). However, there is still a question *how* to do this. Saying it another way, what is the practical approach to classify all inequivalent subalgebras of the algebra ℓ_∞? Before introducing the details of our approach to solving this problem we give a brief account of necessary notions and facts from the general theory of Lie algebras.

Let the symbol L stand for a Lie algebra. Denote by the symbol $[L, L]$ the Lie algebra spanned by all the possible commutators of basis elements of L. Then the Lie algebra N is called a subalgebra of L provided $[N, N] \subset N$. Next, if the relation $[L, N] \subset N$ holds true, then the Lie algebra N is called the ideal of L.

Given two ideals, N_1 and N_2 of the algebra L, the Lie algebra $[N_1, N_2]$ is also the ideal of L. Consider the following sequence of ideals:

$$L^{[0]} = L, \qquad L^{[1]} = [L^{[0]}, L], \ldots, L^{[n]} = [L^{[n-1]}, L], \ldots.$$

If $L^{[n]} = 0$ for some $n > 1$, then the Lie algebra L is called nilpotent. If, in particular, $L^{[1]} = 0$, then L is commutative or Abelian Lie algebra.

Now consider another sequence of ideals of L (the composition series of the Lie algebra L):

$$L^{(0)} = L, \qquad L^{(1)} = [L^{(0)}, L^{(0)}], \ldots, L^{(n)} = [L^{(n-1)}, L^{(n-1)}], \ldots.$$

The algebra L is called solvable if there is $n > 0$ such that $L^{(n)} = 0$. The Lie algebra R is called the radical of L, provided it is the maximal solvable ideal of L containing any other solvable ideal of L.

The Lie algebra L is called semi-simple, provided it does not contain non-zero solvable ideals. Finally, the algebra L is called simple if it contains no ideals different from 0 and L and $L^{[1]} \neq 0$. Evidently, a simple algebra is semi-simple. On the other hand, every semi-simple Lie algebra can be decomposed into a direct sum of simple Lie algebras.

The fundamental Levi–Maltsev theorem (see, e.g., [21, 32]) says that for any Lie algebra L with radical R there exists a semi-simple Lie algebra S such that

$$L = S \inplus R.$$

This relation is called the Levi decomposition of the algebra L; and what is more, the algebra S is called the Levi factor of L. The further details can be found, for example, in [21, 32].

Due to the Levi–Maltsev theorem, the problem of classification of subalgebras of the algebra ℓ_∞ can be divided into three subproblems:

(1) Classification of semi-simple subalgebras.
(2) Classification of solvable subalgebras.
(3) Classification of subalgebras which are semi-direct sums of semi-simple and solvable Lie algebras.

Remarkably, it is possible to complete all of the above classifications for the case of the very general class of PDEs (1.1) starting with one-dimensional subalgebras and increasing step-by-step the dimension of the algebras involved.

One of our principal results presented below is the analogue of the Magadeev theorem for transformation group acting in the space of variables t, x, u. Namely, we prove that the dimension of the maximal invariance algebra of (1.1) is either less or equal to seven or infinite. In the latter case, the invariant equation can be linearized by a point transformation of the space of variables t, x, u.

Our algorithm for group classification in its present form has been suggested in [21, 23]. As the first step, we compute the maximal equivalence group, $\mathcal{E}$, of the class of PDEs (1.1). The equivalence group consists of one-to-one transformations of the space V:

$$\bar{t} = \alpha(t, x, u), \qquad \bar{x} = \beta(t, x, u), \qquad v = \gamma(t, x, u), \qquad \frac{D(\alpha, \beta, \gamma)}{D(t, x, u)} \neq 0, \tag{2.6}$$

which preserve the differential structure of equation (1.1). Note that we use a concept of equivalence group which is different from that employed by Ovsyannikov. Indeed, the group $\mathcal{E}$ does not involve the function F as an extra variable, as it is customary within the Ovsyannikov's classification scheme [2].

Making change of variables (2.6) in equation (1.1) and requiring for the transformed equation to belong to the same class of PDEs as the initial one, namely,

$$v_{\bar{t}} = \Phi(\bar{t}, \bar{x}, v, v_{\bar{x}}, v_{\bar{x}\bar{x}}),$$

one gets the following result (see, e.g., [33]):

Assertion 2.2. *The maximal equivalence group of the class of equations (1.1) is formed by the transformations*

$$\begin{aligned} &\bar{t} = T(t), \qquad \bar{x} = X(t, x, u), \qquad v = U(t, x, u), \\ &T' = \frac{\mathrm{d}T}{\mathrm{d}t} \neq 0, \qquad \frac{D(X, U)}{D(x, u)} \neq 0. \end{aligned} \tag{2.7}$$

Note that the above group contains arbitrary functions, which means that it is the infinite-parameter Lie transformation group.

The equivalence group splits the class of equations (1.1) into conjugacy subclasses, each of them being uniquely characterized by a single representative. So it suffices to provide a representative of each conjugacy class in order to get the complete description of invariant equations. This property is used to simplify the form of infinitesimal operators forming the basis of admissible realizations of Lie algebras. Classification of these realizations is thus reduced to constructing inequivalent realizations of abstract Lie algebras by differential operators in three real variables of the form (2.3). Note that the systematic analysis of realizations of Lie algebras by differential operators in one and two variables was performed by Sophus Lie itself [34–36].

Consider, as an example, one-dimensional Lie algebras generated by operators (2.3). Change of variables (2.7) reduces (2.3) to the form

$$\tilde{v} = \tau T' \partial_{\bar{t}} + (\tau X_t + \xi X_x + \eta X_u)\partial_{\bar{x}} + (\tau U_t + \xi U_x + \eta U_u)\partial_v. \tag{2.8}$$

If $\tau \neq 0$, then choosing a solution of equation $\tau T' = 1$ as T and taking as X, U the fundamental solution of the system of equations

$$\tau X_t + \xi X_x + \eta X_u = 0, \qquad \tau U_t + \xi U_x + \eta U_u = 0, \qquad \frac{D(X, U)}{D(x, u)} \neq 0,$$

we get $\tilde{v} = \partial_{\bar{t}}$.

Now if $\tau = 0$, then $\xi \neq 0$ or $\eta \neq 0$ (otherwise, operator (2.3) vanishes identically). In the case when $\xi \neq 0$, $\eta = 0$, making the change of variables

$$\bar{t} = t, \qquad \bar{x} = u, \qquad v = x,$$

which belongs to the group $\mathcal{E}$, we reduce (2.3) to the form $\tilde{v} = \xi(\bar{t}, \bar{x}, v)\partial_v$. Consequently, we can suppose that $\eta \neq 0$ without any loss of generality. Choosing as X and U non-vanishing identically solutions of the system of PDEs

$$\xi X_x + \eta X_u = 0, \qquad \xi U_x + \eta U_u = 1,$$

we transform (2.8) to become $\tilde{v} = \partial_v$. We summarize the above reasonings in the form of lemma.

Lemma 2.1. *Operator (2.3) is equivalent to one of the canonical operators*

$$v_1 = \partial_t, \qquad v_2 = \partial_u.$$

Thus, there are two $\mathcal{E}$-inequivalent classes of realizations of one-dimensional Lie algebras by operators (2.3). Consequently, there are only two inequivalent classes of PDEs (1.1) admitting one-dimensional Lie algebras. They are easily obtained by integration of the determining equations for the corresponding infinitesimal operators ∂_t, ∂_u.

Theorem 2.1. *There are two inequivalent classes of invariant equations of the form (1.1) admitting one-parameter Lie groups. Below we give the representatives of these classes and the corresponding one-dimensional invariance Lie algebras* A_1*:*

$$u_t = F(x, u, u_x, u_{xx}) : A_1^1 = \langle \partial_t \rangle;$$
$$u_x = F(t, x, u_x, u_{xx}) : A_1^2 = \langle \partial_u \rangle.$$

What is more, if the function F is arbitrary, then the algebras A_1^1 *and* A_1^2 *are maximal in Lie's sense invariance algebras admitted by the corresponding PDEs.*

As we noted in [31], the set of invariant equations (1.1) is naturally split into two classes, $\mathcal{C}_1$ and $\mathcal{C}_2$. Let $A_k = \langle v_1, v_2, \ldots, v_k \rangle$, where

$$v_i = \tau^i \partial_t + \xi^i \partial_x + \eta^i \partial_u, \qquad i = 1, 2, \ldots, k \tag{2.9}$$

be the maximal invariance algebra of equation (1.1). We say that this equation belongs to the first class, $\mathcal{C}_1$, if the functions τ^i $(i = 1, 2, \ldots, k)$ are linearly independent. Otherwise, it belongs to $\mathcal{C}_2$.

According to [31] any nonlinear PDE from the second class $\mathcal{C}_2$ can be mapped into quasi-linear evolution equations by a non-point transformation.

So if one constructs all $\mathcal{E}$-inequivalent equations belonging to the classes $\mathcal{C}_1, \mathcal{C}_2$, then the problem of group classification of the general evolution equation is solved. We provide full calculation details for the first class $\mathcal{C}_1$. In the case of class $\mathcal{C}_2$, we will give outlines of the proofs of the corresponding theorems together with the lists of invariant equations and their maximal invariance algebras.

The obtained list of invariant equations is too large to fit into a single paper. That is why, in order to keep the exposition compact we split the material into two parts. The first part deals with invariant equations from the class $\mathcal{C}_1$ and those equations from $\mathcal{C}_2$ whose invariance algebras are either semi-simple or semi-direct sums of semi-simple and solvable Lie algebras. This paper contains exhaustive classification of these equations. Classification results for PDEs (1.1) invariant under the solvable Lie algebras will be the topic of our subsequent publication.

3. Group classification of equations from $\mathcal{C}_1$

It is the direct consequence of lemma 2.1 that one of the basis operators of the invariance algebra of an equation from $\mathcal{C}_1$ can always be chosen in the from ∂_t. Also according to theorem 2.1, the most general equation from the class $\mathcal{C}_1$ that admits one-dimensional Lie algebra is equivalent to PDE:

$$u_t = F(x, u, u_x, u_{xx}), \qquad F_{u_{xx}} \neq 0. \tag{3.1}$$

The maximal invariance algebra of the above equation reads as $A_1^1 = \langle \partial_t \rangle$.

Now we turn to equations admitting two-dimensional Lie algebras. It so happens that equations from $\mathcal{C}_1$ cannot admit two-dimensional commutative algebras.

Lemma 3.1. *Class* $\mathcal{C}_1$ *does not contain PDEs invariant under two-dimensional commutative Lie algebras.*

Proof. Suppose that the assertion of the lemma does not hold. Then one of the basis operators of the symmetry algebra can be reduced to the form $v_1 = \partial_t$, while the second operator, v_2, is of the generic form (2.3). Inserting these operators into the commutation relation for the two-dimensional commutative Lie algebra, $[v_1, v_2] = 0$, yields that $\tau_t = 0, \xi_t = 0, \eta_t = 0$. Hence it follows that the equation under study belongs to the class $\mathcal{C}_2$. We arrive at the contradiction proving the lemma. □

It follows from the Levi–Maltsev theorem (see, e.g., [32]) that the set of finite-dimensional real Lie algebras consists of solvable algebras and Lie algebras having nontrivial Levi ideal. That is why we proceed now to studying equations from $\mathcal{C}_1$ that admit solvable and semi-simple Lie algebras of symmetry operators (2.8).

3.1. Invariance under solvable Lie algebras

There are two inequivalent two-dimensional solvable Lie algebras

$$A_{2.1} = \langle e_1, e_2\rangle : [e_1, e_2] = 0, \qquad A_{2.2} = \langle e_1, e_2\rangle : [e_1, e_2] = e_2. \tag{3.2}$$

The case of commutative algebra $A_{2.1}$ has already been considered. Turn to the algebra $A_{2.2}$. Without loss of generality we may choose one of the basis operators, say, e_2 to be equal to ∂_t (lemma 2.1).

Taking as e_1 an arbitrary operator of the form (2.3) with $\tau \neq 0$ and inserting e_1 into (3.2) yields $\tau_t = -1, \xi_t = \eta_t = 0$. Consequently, the operator e_1 has the form

$$e_1 = -t\partial_t + \xi(x, u)\partial_x + \eta(x, u)\partial_u, \tag{3.3}$$

where ξ, η are arbitrary smooth functions. If $\xi = \eta = 0$, then we get the realization $\langle -t\partial_t, \partial_t\rangle$. Next, provided $|\xi| + |\eta| \neq 0$, there is a transformation from the group $\mathcal{E}$ which does not alter e_2 and reduce e_1 (3.3) to the form $\tilde{e}_1 = -\bar{t}\partial_{\bar{t}} - v\partial_v$.

Thus, there exist two inequivalent realizations of the Lie algebra $A_{2.2}$

$$\langle -t\partial_t, \partial_t\rangle, \qquad \langle -t\partial_t - u\partial_u, \partial_t\rangle$$

that can be admitted by equations from $\mathcal{C}_1$.

However, if we insert coefficients of the first basis operator of the first realization into the classifying equation (2.4), we get $F = 0$. Consequently, $\mathcal{C}_1$ contains no equations invariant under the first realization of $A_{2.2}$.

A similar analysis of the second realization yields the following class of $A_{2.2}$-invariant PDEs:

$$u_t = F(x, \omega, w), \qquad \omega = u^{-1}u_x, \qquad w = u^{-1}u_{xx}.$$

Consider now three-dimensional solvable Lie algebras. It is a common knowledge that any solvable three-dimensional Lie algebra, $A_3 = \langle e_1, e_2, e_3\rangle$, is either equivalent to one of the two decomposable

$$A_{3.1} = A_{2.1} \oplus A_1 : [e_i, e_j] = 0, \qquad i, j = 1, 2, 3;$$
$$A_{3.2} = A_{2.2} \oplus A_1 : [e_1, e_2] = e_2, \qquad [e_1, e_3] = [e_2, e_3] = 0;$$

or equivalent to one of the seven non-decomposable algebras

$$A_{3.3} : [e_2, e_3] = e_1, \qquad [e_1, e_2] = [e_1, e_3] = 0;$$
$$A_{3.4} : [e_1, e_3] = e_1, \qquad [e_2, e_3] = e_1 + e_2, \qquad [e_1, e_2] = 0;$$
$$A_{3.5} : [e_1, e_3] = e_1, \qquad [e_2, e_3] = e_2, \qquad [e_1, e_2] = 0;$$
$$A_{3.6} : [e_1, e_3] = e_1, \qquad [e_2, e_3] = -e_2, \qquad [e_1, e_2] = 0;$$
$$A_{3.7} : [e_1, e_3] = e_1, \qquad [e_2, e_3] = qe_2, \qquad [e_1, e_2] = 0 \quad (0 < |q| < 1);$$
$$A_{3.8} : [e_1, e_3] = -e_2, \qquad [e_2, e_3] = e_1, \qquad [e_1, e_2] = 0;$$
$$A_{3.9} : [e_1, e_3] = qe_1 - e_2, \qquad [e_2, e_3] = e_1 + qe_2, \qquad [e_1, e_2] = 0 \quad (q > 0)$$

(see, e.g., [37]).

Each of the above algebras contains a two-dimensional commutative subalgebra. By force of lemma 3.1 there are no equations from $\mathcal{C}_1$ which admit one of the above three-dimensional algebras. Since any n-dimensional solvable Lie algebra contains $(n-1)$-dimensional solvable subalgebra, we arrive at the following assertion.

Lemma 3.2. *Any equation from $\mathcal{C}_1$ invariant under solvable Lie algebra of the dimension $n \geqslant 2$ is equivalent to PDE*

$$u_t = F(x, \omega, w), \qquad \omega = u^{-1}u_x, \qquad w = u^{-1}u_{xx}.$$

If F is arbitrary, then the maximal invariance algebra of the above equation is the two-dimensional Lie algebra $\langle -t\partial_t - u\partial_u, \partial_t\rangle$.

3.2. Invariance under semi-simple Lie algebras

The semi-simple Lie algebras of the lowest dimension read [32]

$$so(3) = \langle e_2, e_2, e_3\rangle : [e_1, e_2] = e_3, \qquad [e_1, e_3] = -e_2, \qquad [e_2, e_3] = e_1;$$
$$sl(2, \mathbb{R}) = \langle e_1, e_2, e_3\rangle : [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1.$$

The above algebras do not contain the commutative algebra $A_{2.1}$[40]. However, semi-simple algebras of higher dimension $n > 3$ do contain subalgebras equivalent to $A_{2.1}$ (see, e.g., [32]) and, consequently, cannot be invariance algebras of PDEs from the class $\mathcal{C}_1$. Thus, the only semi-simple Lie algebras that may lead to new invariant equations are the algebras $so(3)$ and $sl(2, \mathbb{R})$.

Consider first the algebra $so(3)$. Inserting $e_1 = \partial_t, e_2, e_3$, where the latter two are of generic form (2.3), into the first two commutation relations of $so(3)$ and integrating the equations obtained yield

$$e_2 = C\cos t\partial_t + (\alpha\cos t + \beta\sin t)\partial_x + (\gamma\cos t + \theta\sin t)\partial_u, \qquad e_3 = [\partial_t, e_2].$$

Here C is an arbitrary non-zero constant, $\alpha = \alpha(x, u), \beta = \beta(x, u), \gamma = \gamma(x, u)$ and $\theta = \theta(x, u)$ are arbitrary real-valued functions. Inserting the obtained expressions for e_2, e_3 into the remaining commutation relation we get $C^2 = -1$. Since this equation has no real solutions, the class $\mathcal{C}_1$ contains no $so(3)$-invariant equations.

Turn now to the algebra $sl(2, \mathbb{R})$. Let the operator e_3 be of the form ∂_t and the operators e_1, e_2 be of generic form (2.3). Inserting the expressions for e_1, e_2, e_3 into the commutation relations and solving the obtained equations we get four $\mathcal{E}$-inequivalent realizations of $sl(2, \mathbb{R})$:

$$\langle 2t\partial_t, -t^2\partial_t, \partial_t\rangle, \qquad \langle 2t\partial_t + x\partial_x, -t^2\partial_t - tx\partial_x, \partial_t\rangle,$$
$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u, \partial_t\rangle, \qquad \langle 2t\partial_t + x\partial_x, -t^2\partial_t + x(x^2 - t)\partial_x, \partial_t\rangle.$$

The first realization cannot be invariance algebra of an equation of the form (1.1).

Requiring invariance of (1.1) under the second realization yields the following system of equations for $F = F(x, u, u_x, u_{xx})$:

$$2F = -u_xF_{u_x} - 2u_{xx}F_{u_{xx}} + xF_x, \qquad xu_x - 2tF = tu_xF_{u_x} + 2tu_{xx}F_{u_{xx}} - txF_x.$$

As the function F is independent of t, we have $xu_x = 0$. Consequently, this realization cannot be admitted by an equation of the form (1.1).

The third realization gives rise to the following system of equations for the function F:

$$2u_{xx}F_{u_{xx}} + u_xF_{u_x} - xF_x = 2F, \qquad 2F_{u_{xx}} + 2xF_{u_x} + x^2F_u = xu_x.$$

This system is compatible. Its general solution is

$$F = x^{-1}uu_x - x^{-2}u^2 + x^{-2}\tilde{F}(\omega, w), \qquad \omega = x^2u_{xx} - 2u, \qquad w = 2u - xu_x.$$

Finally, the last realization yields the following system of determining equations for $F = F(x, u, u_x, u_{xx})$:

$$2u_{xx}F_{u_{xx}} + u_xF_{u_x} - xF_x = 2F, \qquad 6x(u_x + xu_{xx})F_{u_{xx}} + 3x^2u_xF_{u_x} - x^3F_x = -xu_x,$$

whence

$$F = -\tfrac{1}{4}x^{-1}u_x + x^{-3}u_x^{-1}\tilde{F}(u, \omega), \qquad \omega = u_x^{-2}u_{xx} + 3x^{-1}u_x^{-1}.$$

Summarizing the above reasonings, we conclude that the class $\mathcal{C}_1$ contains only two inequivalent classes of PDEs which are invariant under semi-simple Lie algebras of symmetry operators. In both cases, the maximal invariance algebras are isomorphic to $sl(2, \mathbb{R})$.

3.3. Finalizing group classification of equations from $\mathcal{C}_1$

To complete group classification we have to consider equations from $\mathcal{C}_1$ invariant under the Lie algebras ℓ having nontrivial Levi factor and non-zero radical. It is a common knowledge that these Lie algebras should be decomposable into semi-direct sums of semi-simple ℓ_1 and solvable ℓ_2 Lie algebras. Structure of Lie algebras having the Levi factor $sl(2, \mathbb{R})$ is studied in [41]. Since the solvable algebra, ℓ_2, is either one-dimensional or isomorphic to $A_{2.2}$, it follows from the results of [41] that there are no algebras ℓ that are invariance algebras of equations (1.1) belonging to the first class $\mathcal{C}_1$.

Theorem 3.1. *There are, at most, four inequivalent classes of PDEs from $\mathcal{C}_1$ that admit nontrivial invariance algebras. The representative of the first class is (3.1), its maximal symmetry algebra is $\langle \partial_t \rangle$. The representative of the second class is given in lemma 3.2, its maximal invariance algebra being isomorphic to $A_{2.2}$. Two other classes of PDEs are presented below:*

$$u_t = x^{-1}uu_x - x^{-2}u^2 + x^{-2}\tilde{F}(\omega, w), \qquad \omega = x^2u_{xx} - 2u, \qquad w = 2u - xu_x:$$
$$sl^1(2, \mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u, \partial_t \rangle;$$
$$u_t = -\tfrac{1}{4}x^{-1}u_x + x^{-3}u_x^{-1}\tilde{F}(u, \omega), \qquad \omega = u_x^{-2}u_{xx} + 3x^{-1}u_x^{-1}:$$
$$sl^2(2, \mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + x(x^2 - t)\partial_x, \partial_t \rangle.$$

The algebras $sl^1(2, \mathbb{R})$ and $sl^2(2, \mathbb{R})$ are maximal invariance algebras of the corresponding PDEs, provided the functions $\tilde{F}$ are arbitrary.

4. Invariance of equations from $\mathcal{C}_2$ under Lie algebras having nontrivial Levi factor

As we mentioned above, to describe equations (1.1) invariant under Lie algebras having nontrivial Levi factor we need to construct equations, which are invariant under semi-simple Lie algebras. That is why we begin group classification of equations from $\mathcal{C}_2$ by considering realizations of semi-simple Lie algebras by operators (2.3).

4.1. Invariance under semi-simple Lie algebras

It follows from the definition of class $\mathcal{C}_2$ that one of the basis operators of an equation from $\mathcal{C}_2$ can be chosen as ∂_u. Consequently, any invariant equation from $\mathcal{C}_2$ can always be transformed to become

$$u_t = F(t, x, u_x, u_{xx}), \qquad F_{u_{xx}} \neq 0. \tag{4.1}$$

Consider first the semi-simple algebras of the lowest dimension, namely, $so(3)$ and $sl(2, \mathbb{R})$.

Algebra $so(3)$. Let $e_1 = \partial_u$ and let the operators e_2, e_3 be of the form (2.3). Inserting these operators into the first two commutation relations of $so(3)$ and solving the PDEs obtained we get within the equivalence relation $\mathcal{E}$ the following formulae for e_2 and e_3:

$$e_2 = \alpha(t,x)\cos u\partial_x + (\beta(t,x)\cos u + \gamma(t,x)\sin u)\partial_u, \qquad e_3 = [\partial_t, e_2].$$

The remaining commutation relation yields

$$\alpha\beta = 0, \qquad \alpha\gamma_x - \beta^2 - \gamma^2 = 1. \tag{4.2}$$

If $\alpha = 0$, then $\beta^2 + \gamma^2 = -1$ and system (4.2) has no real solutions. In the case $\alpha \neq 0$ we have $\beta = 0$. Making the change of variables from $\mathcal{E}$

$$\bar{t} = t, \qquad \bar{x} = X(t,x)(\alpha X_x = 1), \qquad v = u$$

we get

$$e_2 = \cos u\partial_x + \gamma(t,x)\sin u\partial_u, \qquad e_3 = [\partial_t, e_2],$$

where the function $\gamma = \gamma(t,x)$ is a solution of PDE $\gamma_x = 1 + \gamma^2$, i.e., $\gamma = \tan(x + \varphi(t))$. Making the equivalence transformation

$$\bar{t} = t, \quad \bar{x} = x + \varphi(t), \quad v = u$$

we reduce γ to the form $\tan x$. Consequently, there is only one realization of the algebra $so(3)$ by operators (2.3), namely,

$$so^1(3) = \langle\partial_u, \cos u\partial_x + \tan x\sin u\partial_u, -\sin u\partial_x + \tan x\cos u\partial_u\rangle.$$

Inserting the coefficients of the above operators into the classifying equation (2.4) yields the following system of PDEs for the function $F = F(t, x, u_x, u_{xx})$:

$$u_x F - \big(\sec^2 x + u_x^2\big)F_{u_x} - \big(2\tan x\sec^2 x - u_x^2\tan x + 3u_x u_{xx}\big)F_{u_{xx}} = 0,$$
$$u_x\tan x F_{u_x} + \big(2u_x\sec^2 x + u_x^3 + u_{xx}\tan x\big)F_{u_{xx}} + F_x - \tan x F = 0.$$

The general solution of the above system reads

$$F = \sqrt{\sec^2 x + u_x^2}\,\tilde{F}(t,\omega),$$
$$\omega = \big(u_{xx}\cos x - (2 + u_x^2\cos^2 x)u_x\sin x\big)\big(1 + u_x^2\cos^2 x\big)^{-3/2}. \tag{4.3}$$

What is more, the algebra $so(3)$ is the maximal invariance algebra of the corresponding equation, provided $\tilde{F}$ is arbitrary.

Lemma 4.1. *Any* $so(3)$*-invariant equation (4.1) from* $\mathcal{C}_2$ *is equivalent to PDE*

$$u_t = \sqrt{\sec^2 x + u_x^2}\,\tilde{F}(t,\omega), \quad \omega = \big(u_{xx}\cos x - \big(2 + u_x^2\cos^2 x\big)u_x\sin x\big)\big(1 + u_x^2\cos^2 x\big)^{-3/2}.$$

Its maximal symmetry algebra is

$$so^1(3) = \langle\partial_u, \cos u\partial_x + \tan x\sin u\partial_u, -\sin u\partial_x + \tan x\cos u\partial_u\rangle.$$

Algebra $sl(2,\mathbb{R})$. Let $e_3 = \partial_u$ and e_1, e_2 be of the form (2.3). In order to satisfy the commutation relations of the algebra $sl(2,\mathbb{R})$, the operators e_1, e_2, e_3 have to be of the form

$$e_2 = (\alpha u + \beta)\partial_x + (-u^2 + \gamma u + \theta)\partial_u, \qquad e_1 = -[\partial_u, e_2], \qquad e_3 = \partial_u,$$

where $\alpha = \alpha(t,x)$, $\beta = \beta(t,x)$, $\gamma = \gamma(t,x)$, $\theta = \theta(t,x)$ are solutions of the system of PDEs

$$2\beta = -\alpha\beta_x - \alpha\gamma + \beta\alpha_x, \qquad 4\theta = -\alpha\theta_x - \gamma^2 + \beta\gamma_x. \tag{4.4}$$

If the function α does not vanish identically, then using the transformation

$$\bar{t} = t, \qquad \bar{x} = X(t,x), \qquad v = u + U(t,x),$$

where X, U are solutions of PDEs $\alpha X_x = X$, $XU = \beta X_x$, we simplify e_1, e_2

$$e_2 = xu\partial_x + (-u^2 + \gamma u + \theta)\partial_u, \qquad e_1 = -[\partial_u, e_2], \qquad e_3 = \partial_u.$$

In this case, system (4.4) takes the form $x\gamma = 0, 4\theta = -x\theta_x - \gamma^2$, whence it follows that $\gamma = 0, \theta = \mu(t)x^{-4}$. Provided $\mu = 0$, we have the following realization of the algebra $sl(2, \mathbb{R})$:

$$\langle 2u\partial_u - x\partial_x, -u^2\partial_u + xu\partial_x, \partial_u\rangle.$$

If $\mu \neq 0$, then making the transformation

$$\bar{t} = t, \qquad \bar{x} = |\mu|^{-\frac{1}{4}}x, \qquad v = u,$$

we arrive at the two realizations of the algebra $sl(2, \mathbb{R})$:

$$\langle 2u\partial_u - x\partial_x, (x^{-4} - u^2)\partial_u + xu\partial_x, \partial_u\rangle, \qquad \langle 2u\partial_u - x\partial_x, -(x^{-4} + u^2) + xu\partial_x, \partial_u\rangle.$$

Given the condition $\alpha = 0$, we get from (4.4) the following realization of $sl(2, \mathbb{R})$:

$$\langle 2u\partial_u, -u^2\partial_u, \partial_u\rangle.$$

However, this realization cannot be invariance algebra of an equation of the form (4.1), while the preceding ones give rise to the three classes of $sl(2, \mathbb{R})$-invariant equations.

Lemma 4.2. *There are only three inequivalent classes of equations from $\mathcal{C}_2$ whose invariance algebras are isomorphic to $sl(2, \mathbb{R})$. Below we present the canonical forms of these equations together with their maximal invariance algebras:*

$$u_t = xu_x\tilde{F}(t, \omega), \qquad \omega = x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2} :$$
$$sl^3(2, \mathbb{R}) = \langle 2u\partial_u - x\partial_x, -u^2\partial_u + xu\partial_x, \partial_u\rangle;$$
$$u_t = x^{-2}\sqrt{4 + x^6u_x^2}\,\tilde{F}(t, \omega), \qquad \omega = \left(4 + x^6u_x^2\right)^{-3/2}\left(x^4u_{xx} + 5x^3u_x + \tfrac{1}{2}x^9u_x^3\right) :$$
$$sl^4(2, \mathbb{R}) = \langle 2u\partial_u - x\partial_x, (x^{-4} - u^2)\partial_u + xu\partial_x, \partial_u\rangle;$$
$$u_t = x^{-2}\sqrt{|x^6u_x^2 - 4|}\,\tilde{F}(t, \omega), \qquad \omega = (x^6u_x^2 - 4)^{-3/2}\left(x^4u_{xx} + 5x^3u_x - \tfrac{1}{2}x^9u_x^3\right) :$$
$$sl^5(2, \mathbb{R}) = \langle 2u\partial_u - x\partial_x, -(x^{-4} + u^2)\partial_u + xu\partial_x, \partial_u\rangle.$$

Semi-simple algebras of the dimension higher than 3. According to the general classification of semi-simple Lie algebras [32], the next possible dimension of a semi-simple Lie algebra is 6. There are four non-isomorphic semi-simple algebras of the dimension 6, $so(4)$, $so(3, 1)$, $so(2, 2)$ and $so^*(4)$.

It is well known that $so(4) = so(3) \oplus so(3)$, $so^*(4) \sim so(3) \oplus sl(2, \mathbb{R})$. Consequently, we can utilize the results of classification of realizations of $so(3)$ and $sl(2, \mathbb{R})$.

To classify $so(4)$-invariant equations (1.1) we need to construct all realizations of $so(3)$ by operators (2.3), which commute with the basis operators of the realization $so^1(3)$. It is straightforward to verify that such realizations do not exist. Similarly, to classify $so^*(4)$-invariant equations (1.1) we have to construct all realizations of $sl(2, \mathbb{R})$ by operators (2.3), which commute with the basis operators of the realization $so^1(3)$. The only realization of $sl(2, \mathbb{R})$ obeying the above constraint is equivalent to $\langle 2t\partial_t, -t^2\partial_t, \partial_t\rangle$. The latter cannot be invariance algebra of an equation from $\mathcal{C}_2$. Consequently, there are no equations belonging to the class $\mathcal{C}_2$, which are invariant under the algebras isomorphic to $so(4)$ and $so^*(4)$.

The same assertion holds for the algebra $so(3, 1)$. Indeed, the algebra $so(3, 1)$ has the Cartan decomposition $\langle e_1, e_2, e_3\rangle \dotplus \langle N_1, N_2, N_3\rangle$, where $\langle e_1, e_2, e_3\rangle = so(3)$, $[e_i, N_j] =$

$\sum_{l=1}^{3} \varepsilon_{ijl} N_l$, $[N_i, N_j] = -\sum_{l=1}^{3} \epsilon_{ijl} e_l$, $i, j, l = 1, 2, 3$ and ε_{ijl} is the anti-symmetric third-order tensor with $\varepsilon_{123} = 1$. Taking $so(3) = so^1(3)$ after simple calculations we get the forms of operators N_1, N_2, N_3:

$$N_1 = \cos u \partial_u, \quad N_2 = -\sec u \cos x \partial_x + \sin u \sin x \partial_u, \quad N_3 = \sec u \sin x \partial_x + \sin u \cos x \partial_u.$$

Inserting the coefficients of the operator N_1 into the classifying equation (2.4) yields $F_{u_{xx}} = 0$, whence it follows that the realization obtained cannot be invariance algebra of PDE (1.1).

In order to classify realizations of the algebra $so(2, 2)$, we make use of the fact that $so(2, 2) \sim sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. So that we can choose $\langle e_i, \bar{e}_i \,|i = 1, 2, 3\rangle$ as the basis of $so(2, 2)$. Here $\langle e_1, e_2, e_3\rangle = sl(2, \mathbb{R})$, $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3\rangle = sl(2, \mathbb{R})$ with $[e_i, \bar{e}_j] = 0\,(i, j = 1, 2, 3)$. Choosing as e_i $(i = 1, 2, 3)$ the basis operators of one of the realizations $sl^k(2, \mathbb{R})$ $(k = 1, 2, \ldots, 5)$ and analysing the commutation relations of $so(2, 2)$ we come to the conclusion that the class of operators (2.3) does not contain realizations of the algebra $so(2, 2)$ which are admitted by a nonlinear equation of the form (1.1).

The same assertion holds true for the semi-simple algebras $sl(3, \mathbb{R})$, $su(3)$ and $su(2, 1)$, which are eight dimensional (there are no seven-dimensional semi-simple algebras). Actually, this is true for any semi-simple algebra of the dimension $n > 3$. Indeed, there are four basic types of classical simple Lie algebras over the field of real numbers:

- Type $A_{n-1}(n > 1)$ contains four real forms of the algebra $sl(n, \mathbb{C})$: $su(n)$, $sl(n, \mathbb{R})$, $su(p, q)$ $(p + q = n,\ p \geqslant q)$, $su^*(2n)$.
- Type $D_n(n > 1)$ contains three real forms of the algebra $so(2n, \mathbb{C})$: $so(2n)$, $so(p, q)$ $(p + q = 2n,\ p \geqslant q)$, $so^*(2n)$.
- Type $B_n(n > 1)$ contains two real forms of the algebra $so(2n+1, \mathbb{C})$: $so(2n+1)$, $so(p, q)$ $(p + q = 2n + 1,\ p > q)$.
- Type $C_n(n > 1)$ contains three real forms of the algebra $sp(n, \mathbb{C})$: $sp(n)$, $sp(n, \mathbb{R})$, $sp(p, q)$ $(p + q = n,\ p > q$.

As $su^*(4) \sim so(5, 1)$, and furthermore the algebra $so(5, 1)$ contains $so(4)$ as a subalgebra, the class of operators (2.3) does not contain realizations of the algebras A_{n-1} $(n > 1)$ and D_n $(n > 1)$, which differ from $sl^k(2, \mathbb{R})$ $(k = 1, 2, \ldots, 5)$.

The same assertion holds for the algebras B_n $(n > 1)$ and C_n $(n \geqslant 1)$ as well. Indeed, algebras of the type B_n with $n \geqslant 2$ contain $so(4)$ and $so(3, 1)$. What is more, $sp(2, \mathbb{R}) \sim so(3, 2)$ and $so(3, 1)$ is the subalgebra of $so(3, 2)$, $sp(1, 1) \sim so(4, 1)$ and $so(3, 1)$ is the subalgebra of $so(4, 1)$, $sp(2) \sim so(5)$ and $so(4)$ is the subalgebra of $so(5)$.

It remains to analyse the exceptional semi-simple Lie algebras $G_1, G_2, F_4, E_6, E_7, E_8$. We consider the case of the algebra G_2 only, the remaining algebras are handled in the same way.

A Lie algebra of the type G_2 contains the compact real form g_2 and the non-compact real form g_2'. Since $g_2 \cap g_2' \sim su(2) \oplus su(2) \sim so(4)$, the class of operators (2.3) does not contain realizations of the algebras g_2 and g_2', which would be invariance algebras of equation (1.1). The same assertion holds true for the remaining exceptional semi-simple Lie algebras G_1, F_4, E_6, E_7, E_8.

Let us summarize the above reasonings in the form of theorem.

Theorem 4.1. *Any equation from the class $\mathcal{C}_2$, whose invariance algebra is semi-simple, is equivalent to one of the equations given in lemmas 4.1 and 4.2.*

4.2. Invariance under the algebras having nontrivial Levi factor

Now we utilize the results of classification of inequivalent equations (1.1) admitting semi-simple symmetry algebras to describe PDEs (1.1) whose symmetry algebras admit Levi decomposition. The class of Lie algebras that admit Levi decomposition splits into two non-intersecting subclasses:

- subclass of Lie algebras which are decomposable into direct sums of semi-simple and solvable Lie algebras and
- subclass of Lie algebras which are semi-direct sums of a Levi factor and non-zero radical.

4.2.1. Invariance under direct sum of semi-simple and solvable Lie algebras. To describe equations whose symmetry algebras are direct sums semi-simple and solvable Lie algebras we can utilize the explicit forms of realizations of semi-simple algebras constructed in the previous subsections. What is more, we need to consider realizations of semi-simple algebras belonging both to the class $\mathcal{C}_1$ and to the class $\mathcal{C}_2$.

Consider the case of the $sl^1(2,\mathbb{R})$-invariant equation. We look for possible extensions of the realization $sl^1(2,\mathbb{R})$ by operators (2.3) which commute with its basis operators. Analysis of the commutativity conditions yields the general form of the additional symmetry operators

$$v = C_1 x\partial_x + (C_2 + 2C_1 u)\partial_u, \tag{4.5}$$

where C_1, C_2 are arbitrary constants. So we need to describe all possible solvable Lie algebras that have basis operators (4.5). Skipping intermediate calculations we formulate the final result. The list of isomorphic solvable Lie algebras realized by operators (4.5) is formed by the two one-dimensional algebras $\langle\partial_u\rangle$, $\langle x\partial_x + 2u\partial_u\rangle$ and one two-dimensional algebra $L_2 = \langle\partial_u, x\partial_x + 2u\partial_u\rangle$. Note that the latter is isomorphic to $A_{2.2}$.

Inserting the coefficients of the above operators into the classifying equation and solving the resulting equations yield the forms of the unknown functions F in the corresponding invariant equations:

(1) Algebra $sl^1(2,\mathbb{R}) \oplus \langle\partial_u\rangle$

$$u_t = \tfrac{1}{4}u_x^2 + x^{-2}\tilde{F}(\omega), \qquad \omega = x^2 u_{xx} - x u_x;$$

(2) Algebra $sl^1(2,\mathbb{R}) \oplus \langle x\partial_x + 2u\partial_u\rangle$

$$u_t = x^{-1}uu_x - x^{-2}u^2 + x^{-2}(2u - xu_x)^2\tilde{F}(\omega), \qquad \omega = (x^2u_{xx} - 2u)(2u - xu_x)^{-1};$$

(3) Algebra $sl^1 2,(\mathbb{R}) \oplus \langle\partial_u, x\partial_x + 2u\partial_u\rangle$

$$u_t = mx^2u_{xx}^2 - 2mxu_xu_{xx} + \left(m + \tfrac{1}{4}\right)u_x^2, \qquad m \neq 0.$$

Under arbitrary $\tilde{F}$ the given algebras are maximal in Lie's sense invariance algebras of the corresponding equations.

Analogously, we prove that the basis operators of a solvable Lie algebra which commute with the basis operators of $sl^2(2,\mathbb{R})$ are necessarily of the form

$$v = \eta(u)\partial_u. \tag{4.6}$$

Further computations show that the maximal dimension of a solvable Lie algebra having basis operators (4.6) is 2. There are two inequivalent realizations, one-dimensional $L_1 = \langle\partial_u\rangle$ and two-dimensional $L_2 = \langle -u\partial_u, \partial_u\rangle$ with $L_2 \sim A_{2.2}$. Solving the corresponding classifying equations yields the forms of the right-hand sides of invariant equations (1.1):

(1) Algebra $sl^2(2,\mathbb{R}) \oplus \langle\partial_u\rangle$

$$u_t = -\tfrac{1}{4}x^{-1}u_x + x^{-3}u_x^{-1}\tilde{F}(\omega), \qquad \omega = u_x^{-2}u_{xx} + 3x^{-1}u_x^{-1};$$

(2) Algebra $sl^2(2,\mathbb{R}) \oplus \langle -u\partial_u, \partial_u\rangle$

$$u_t = -\tfrac{1}{4}x^{-1}u_x + mx^{-3}u_x^{-1}\left(u_x^{-2}u_{xx} + 3x^{-1}u_x^{-1}\right)^{-2}, \qquad m \neq 0.$$

Under arbitrary $\tilde{F}$ and m the given algebras are maximal in Lie's sense invariance algebras of the corresponding equations.

The most general form of operators (2.3) commuting with the basis operators of $sl^3(2,\mathbb{R})$ is

$$v = \tau(t)\partial_t + \xi(t)x\partial_x. \tag{4.7}$$

Our analysis shows that there exist only four $\mathcal{E}$-inequivalent realizations of solvable Lie algebras by operators (4.7), which are invariance algebras of equations of the form (1.1). Namely, there are two one-dimensional realizations, $\langle\partial_t\rangle$, $\langle tx\partial_x\rangle$, and two two-dimensional realizations, $\langle -t\partial_t - mx\partial_x, \partial_t\rangle (m\in\mathbb{R})$, $\langle t\partial_t, tx\partial_x\rangle$. Below we list the corresponding invariant equations:

(1) Algebra $sl^3(2,\mathbb{R})\oplus\langle\partial_t\rangle$

$$u_t = xu_x\tilde{F}(\omega), \qquad \omega = x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2};$$

(2) Algebra $sl^3(2,\mathbb{R})\oplus\langle tx\partial_x\rangle$

$$u_t = \frac{xu_x}{4t}\ln\left(x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2}\right) + xu_x\tilde{F}(t);$$

(3) Algebra $sl^3(2,\mathbb{R})\oplus\langle -t\partial_t - mx\partial_x, \partial_t\rangle$

$$u_t = \lambda xu_x\left(x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2}\right)^{1/4m}, \qquad \lambda\neq 0, \qquad m\neq 0, \pm\tfrac{3}{4};$$

(4) Algebra $sl^3(2,\mathbb{R})\oplus\langle t\partial_t, tx\partial_x\rangle$

$$u_t = \frac{xu_x}{4t}\ln\left(x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2}\right) + \frac{\lambda xu_x}{t}, \qquad \lambda\in\mathbb{R}.$$

Under arbitrary $\tilde{F}$, m and λ the given algebras are maximal in Lie's sense invariance algebras of the corresponding equations.

A similar analysis of extensions of the realizations algebras $sl^4(2,\mathbb{R})$, $sl^5(2,\mathbb{R})$ and $so^1(3)$ yields three more invariant equations. Below we give the right-hand sides of invariant equations and their maximal invariance algebras:

$$sl^4(2,\mathbb{R})\oplus\langle\partial_t\rangle : u_t = x^{-2}\sqrt{4+x^6u_x^2}\,\tilde{F}(\omega),$$
$$\omega = \left(4+x^6u_x^2\right)^{-\frac{3}{2}}\left(x^4u_{xx} + 5x^3u_x + \tfrac{1}{2}x^9u_x^3\right);$$
$$sl^5(2,\mathbb{R})\oplus\langle\partial_t\rangle : u_t = x^{-2}\sqrt{\left|x^6u_x^2 - 4\right|}\,\tilde{F}(\omega),$$
$$\omega = \left|x^6u_x^2 - 4\right|^{-\frac{3}{2}}\left(x^4u_{xx} + 5x^3u_x - \tfrac{1}{2}x^9u_x^3\right);$$
$$so^1(3)\oplus\langle\partial_t\rangle : u_t = \sqrt{\sec^2 x + u_x^2}\,\tilde{F}(\omega),$$
$$\omega = \left(u_{xx}\cos x - \left(2+u_x^2\cos^2 x\right)u_x\sin x\right)\left(1+u_x^2\cos^2 x\right)^{-3/2}.$$

4.2.2. Invariance under semi-direct sum of semi-simple and solvable Lie algebras. To perform classification of equations from $\mathcal{C}_2$ whose invariance algebra are isomorphic to semi-direct sum of semi-simple and solvable Lie algebras we need to apply a more sophisticated strategy. It is based on the well-known fact of the group analysis of differential equations, which is the higher the dimension of the invariance algebra admitted by PDE (1.1) the less arbitrary is the function F. So at some point, instead of an arbitrary function of five variables t, x, u, u_x, u_{xx} equation (1.1) would contain an arbitrary function of one variable or even arbitrary constants. When this is the case, we apply the Ovsyannikov classification method

[2], since the determining equations split into over-determined systems of PDEs that can be effectively integrated.

So our approach to classification of equations invariant under semi-direct sum of semi-simple and solvable Lie algebras consists of the two major steps. Firstly, utilizing the results of classification of lower dimensional Lie algebras that can be decomposed into semi-direct sum of Levi factor and solvable radical [41] we describe all invariant equations containing arbitrary functions of four, three, two, one arguments and/or arbitrary constants. The second step is utilizing the Ovsyannikov's approach to classify those equations that have either arbitrary functions of one variable or arbitrary parameters. This will provide complete classification of equation (1.1) within the considered class of Lie algebras.

We prove that in order to implement the first step of our method it suffices to consider

(1) Algebras which are semi-simple sums of semi-simple and solvable Lie algebras of the dimension $n \leqslant 6$, if the Levi factor is isomorphic to $so(3)$.
(2) Algebras which are semi-simple sums of semi-simple and solvable Lie algebras of the dimension $n \leqslant 5$, if the Levi factor is isomorphic to $sl(2, \mathbb{R})$.

Further analysis shows that without any loss of generality we can restrict our considerations to the Lie algebras $sl(2, \mathbb{R}) \Subset A_{2.1}$, $so(3) \Subset A_{3.1}$.

After completing the first step, we apply the Ovsyannikov method to finalize the classification. In addition, we utilize this method to complete group classification of invariant equations having arbitrary functions of one variable or arbitrary constants obtained in the previous subsection.

In the following, we use the list of non-isomorphic four-dimensional solvable Lie algebras obtained in [37]. This list is formed by ten decomposable algebras $A_{3.i} \oplus A_1$ $(i = 1, 2, \ldots, 9), 2A_{2.2} = A_{2.2} \oplus A_{2.2}$ and ten non-decomposable algebras $A_{4.i} = \langle e_1, e_2, e_3, e_4\rangle$ $(i = 1, 2, \ldots, 10)$ (we give only non-zero commutation relations only):

$$A_{4.1}: [e_2, e_4] = e_1, \quad [e_3, e_4] = e_2;$$
$$A_{4.2}: [e_1, e_4] = qe_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3, \quad q \neq 0;$$
$$A_{4.3}: [e_1, e_4] = e_1, \quad [e_3, e_4] = e_2;$$
$$A_{4.4}: [e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \quad [e_3, e_4] = e_2 + e_3;$$
$$A_{4.5}: [e_1, e_4] = e_1, \quad [e_2, e_4] = qe_2, \quad [e_3, e_4] = pe_3, \quad -1 \leqslant p \leqslant q \leqslant 1, \quad pq \neq 0;$$
$$A_{4.6}: [e_1, e_4] = qe_1, \quad [e_2, e_4] = pe_2 - e_3, \quad [e_3, e_4] = e_2 + pe_3, \quad q \neq 0, \quad p \geqslant 0;$$
$$A_{4.7}: [e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3;$$
$$A_{4.8}: [e_2, e_3] = e_1, \quad [e_1, e_4] = (1+q)e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = qe_3, \quad |q| \leqslant 1;$$
$$A_{4.9}: [e_2, e_3] = e_1, \quad [e_1, e_4] = 2qe_1, \quad [e_2, e_4] = qe_2 - e_3, \quad [e_3, e_4] = e_2 + qe_3, \quad q \geqslant 0;$$
$$A_{4.10}: [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1.$$

We provide full calculation details for the case of the algebra $sl^1(2, \mathbb{R}) \Subset A_{2.1}$. The algebra $so(3) \Subset A_{3.1}$ is handled in the same way.

Let $sl(2, \mathbb{R}) = \langle e_1, e_2, e_3\rangle, A_{2.1} = \langle e_4, e_5\rangle$. Then the basis elements of $sl^1(2, \mathbb{R})$ are $\mathcal{E}$-equivalent to $e_1 = 2t\partial_t + x\partial_x$, $e_2 = -t^2\partial_t - tx\partial_x + x^2\partial_u$, $e_3 = \partial_t$. The remaining non-zero commutation relations of the algebra $sl^1(2, \mathbb{R}) \Subset A_{2.1}$ read

$$[e_1, e_4] = e_4, \qquad [e_1, e_5] = -e_5, \qquad [e_2, e_5] = e_4, \qquad [e_3, e_4] = e_5. \tag{4.8}$$

Inserting operators e_4, e_5 of the form (2.3) into (4.8) and solving the resulting equations we obtain the four inequivalent realizations of the algebra $sl^1(2, \mathbb{R}) \Subset A_{2.1}$, basis operators of

$A_{2.1}$ having the form

(1) $e_4 = t\partial_x + 2tx^{-1}u\partial_u, \qquad e_5 = \partial_x + 2x^{-1}u\partial_u;$

(2) $e_4 = t\partial_x + (tx^{-1}u - x)\partial_u, \qquad e_5 = \partial_x + x^{-1}u\partial_u;$

(3) $e_4 = tx^{-1}\partial_u, \qquad e_5 = x^{-1}\partial_u;$

(4) $e_4 = (tu + x^2)\partial_x + (2ux + 2tx^{-1}u^2)\partial_u, \qquad e_5 = u\partial_x + 2x^{-1}u^2\partial_u.$

However, only the second and the third realization give rise to the symmetry algebras of equations of the form (1.1). The corresponding invariant equations are

$sl^1(2,\mathbb{R}) \Subset \langle t\partial_x + (tx^{-1}u - x)\partial_u, \partial_x + x^{-1}u\partial_u\rangle$:

$$u_t = \lambda u_{xx} + 2\lambda x^{-2}u - 2\lambda x^{-1}u_x + x^{-1}uu_x - x^{-2}u^2, \qquad \lambda \neq 0;$$

$sl^1(2,\mathbb{R}) \Subset \langle (tu + x^2)\partial_x + 2(xu + tx^{-1}u^2)\partial_u, u\partial_x + 2x^{-1}u^2\partial_u\rangle$:

$$u_t = x^{-1}uu_x - x^{-2}u^2 + \lambda x^{-2}(2u - xu_x)(x^2u_{xx} + 2u - 2xu_x)^{-1}, \qquad \lambda \neq 0.$$

Note that the five-dimensional Lie algebras presented above are maximal in Lie's sense.

Analysis of the realizations $sl^2(2,\mathbb{R})$, $sl^4(2,\mathbb{R})$, $sl^5(2,\mathbb{R})$ shows that they do cannot be extended up to a realization of the algebra $sl(2,\mathbb{R}) \Subset A_{2.1}$. Similarly, the realization $so^1(3)$ cannot be extended up to a realization of the algebra $so^1(3) \Subset A_{3.1}$.

The realization $sl^3(2,\mathbb{R})$ does yield new realizations of the algebra $sl(2,\mathbb{R}) \Subset A_{2.1}$. We give these below together with the corresponding invariant equations:

$$sl^3(2,\mathbb{R}) \Subset \langle -\partial_x + x^{-1}u\partial_u, x^{-1}\partial_u\rangle : u_t = x^{-1}(xu_{xx} + 2u_x)^{1/3}F(t). \tag{4.9}$$

$$sl^3(2,\mathbb{R}) \Subset \langle x^2u\partial_x, x^2\partial_x\rangle : u_t = x^3u_x^2(xu_{xx} + 2u_x)^{-1/3}F(t). \tag{4.10}$$

However these five-dimensional Lie algebras are not maximal. To find the most extensive symmetry algebras we apply the infinitesimal Lie algorithm directly. First of all, using the fact that $F(t) \neq 0$ we can make the change of variables

$$\bar{t} = \int F(t)\,\mathrm{d}t, \qquad \bar{x} = x, \qquad v = u$$

and get $F \equiv 1$. Next, utilizing the Lie's algorithm we obtain that the maximal invariance algebra of PDE (4.9) with $F = 1$ is the seven-dimensional Lie algebra

$$sl^3(2,\mathbb{R}) \Subset \left\langle \partial_t, x\partial_x + \tfrac{4}{3}t\partial_t, x^{-1}\partial_u, -\partial_x + x^{-1}u\partial_u\right\rangle.$$

This algebra is isomorphic to the Lie algebra $sl(2,\mathbb{R}) \Subset A_{4.5}$ with $q = 1$, $p = \frac{4}{3}$. The maximal invariance algebra admitted by (4.10) with $F = 1$ is the seven-dimensional Lie algebra

$$sl^3(2,\mathbb{R}) \Subset \left\langle \partial_t, \tfrac{4}{3}t\partial_t - x\partial_x, x^2u\partial_x, x^2\partial_x\right\rangle.$$

It is isomorphic to the algebra $sl(2,\mathbb{R}) \Subset A_{4.5}$ with $q = 1$, $p = \frac{4}{3}$.

Now we proceed to finalizing group classification of the equations obtained in the previous subsection. Since these equations contain arbitrary functions of at most one variable, we can directly apply the Ovsyannikov classification method.

Consider, as an example, the case of realization $sl^1(2,\mathbb{R}) \oplus \langle\partial_u\rangle$. Inserting the function

$$F = \tfrac{1}{4}u_x^2 + x^{-2}\tilde{F}(\omega), \qquad \omega = x^2u_{xx} - xu_x$$

into the classifying equation (2.4) and splitting the obtained relation by the powers of the independent variable u_x yields the following over-determined system of PDEs for τ, ξ, η

and $\tilde{F}$:

$$(x^{-2}(2x^{-1}\xi + \eta_u - 2\xi_x)\omega - x^{-1}\eta_x + \eta_{xx})\tilde{F}_\omega = x^{-2}(\eta_u - \tau_t + 2x^{-1}\xi)\tilde{F},$$
$$(3x^{-2}\xi_u\omega + \xi_{xx} + x^{-1}\xi_x - x^{-2}\xi - 2\eta_{xu})\tilde{F}_\omega = x^{-2}\xi_u\tilde{F} + \tfrac{1}{2}\eta_x + \xi_t, \qquad (4.11)$$
$$(2\xi_{xu} - \eta_{uu} + 2x^{-1}\xi_u)\tilde{F}_\omega = \tfrac{1}{4}\eta_u + \tfrac{1}{4}\tau_t - \tfrac{1}{2}\xi_x, \qquad \xi_{uu}\tilde{F}_\omega = -\tfrac{1}{4}\xi_u.$$

If the function $\tilde{F}$ (4.11) is arbitrary, then the realization $sl^1(2,\mathbb{R}) \oplus \langle\partial_u\rangle$ is the maximal symmetry algebra of the corresponding equation.

It follows from the last two equations from (4.11) that either $\tilde{F}$ is linear function of ω or

$$\xi_u = \eta_{uu} = 0, \qquad 2\xi_x - \eta_u - \tau_t = 0. \qquad (4.12)$$

Provided $\tilde{F} = \lambda\omega + C, \lambda \neq 0, C \in \mathbb{R}$, then the maximal invariance algebra is infinite dimensional. In the case when $C \neq 3\lambda$ it is formed by the basis elements of the realization $sl^1(2,\mathbb{R}) \oplus \langle\partial_u\rangle$ and by the infinite set of operators

$$v_\infty = \alpha(t,x)\exp\left(\frac{u}{4\lambda}\right)\partial_u,$$

where $\alpha = \alpha(t,x)$ is an arbitrary solution of the equation

$$\alpha_t = \lambda\alpha_{xx} - \lambda x^{-1}\alpha_x - \frac{C}{4\lambda}x^{-2}\alpha.$$

In the case when $C = 3\lambda$, the corresponding PDE admits two more symmetry operators $t\partial_x + 2(\lambda x^{-1}t - x)\partial_u$ and $\partial_x + 2\lambda x^{-1}\partial_u$.

However, the change of variables from $\mathcal{E}$

$$\bar{t} = t, \qquad \bar{x} = x, \qquad u = 4\lambda\ln|v|, \qquad v = v(\bar{t},\bar{x})$$

reduces the equation under study to a linear heat conductivity equation

$$v_{\bar{t}} = \lambda v_{\bar{x}\bar{x}} - \lambda x^{-1}v_{\bar{x}} + \frac{C}{4\lambda}\bar{x}^{-2}v.$$

Consequently, the equation in question is equivalent to linear PDE and therefore is excluded from further consideration.

If the function $\tilde{F}$ is a nonlinear function of ω, i.e., $\tilde{F}_{\omega\omega} \neq 0$, then taking into account (4.12) we derive from (4.11) the system of two equations for the functions τ, $\xi = \xi(t,x)$, $\eta = (2\xi_x - \tau_x)u + \theta(t,x)$ and $\tilde{F}$:

$$(x^{-2}(2x^{-1}\xi + \eta_u - 2\xi_x)\omega - x^{-1}\eta_x + \eta_{xx})\tilde{F}_\omega = x^{-2}(\eta_u - \tau_t + 2x^{-1}\xi)\tilde{F} + \eta_t,$$
$$(2\eta_{xu} - \xi_{xx} - x^{-1}\xi_x + x^{-2}\xi)\tilde{F}_\omega = -\tfrac{1}{2}\eta_x - \xi_t. \qquad (4.13)$$

Analysis of the first equation yields the following admissible forms of the function $\tilde{F}$:

$$\tilde{F} = \lambda\exp(p\omega) + m, \quad \lambda p \neq 0, \quad m \in \mathbb{R};$$
$$\tilde{F} = \lambda\ln|\omega + b| + m, \quad \lambda \neq 0, \quad b, m \in \mathbb{R};$$
$$\tilde{F} = \lambda|\omega + b|^p + m, \quad \lambda p \neq 0, \quad p \neq 1, \quad b, m \in \mathbb{R}.$$

Inserting these expressions into (4.13) shows that extension of symmetry algebra is only possible when $\tilde{F} = \lambda\omega^2$. However, the maximal invariance algebra of this equation has already been obtained earlier in this subsection.

A similar analysis of the remaining invariant equations from subsection 4.2.1 yields the following results:

- The only extension of the realization $sl^2(2,\mathbb{R}) \oplus \langle\partial_u\rangle$, which is invariance algebra of an equation of the form (1.1), is the realization $sl^2(2,\mathbb{R}) \oplus \langle -u\partial_u, \partial_u\rangle$.

- The only extension of the algebra $sl^3(2,\mathbb{R}) \oplus \langle tx\partial_x\rangle$, which is invariance algebra of an equation of the form (1.1), is the realization $sl^3(2,\mathbb{R}) \oplus \langle t\partial_t, tx\partial_x\rangle$.
- The list of possible extensions of the algebra $sl^3(2,\mathbb{R}) \oplus \langle\partial_t\rangle$, which are invariance algebras of equations of the form (1.1), is exhausted by the following algebras:

(1) $sl^3(2,\mathbb{R}) \oplus \langle -t\partial_t - mx\partial_x, \partial_t\rangle$,
(2) $sl^3(2,\mathbb{R}) \oplus \langle t\partial_t, tx\partial_x\rangle$,
(3) $sl^3(2,\mathbb{R}) \Subset \langle \partial_t, x\partial_x + \frac{4}{3}t\partial_t, x^{-1}\partial_u, -\partial_x + x^{-1}u\partial_u\rangle$,
(4) $sl^3(2,\mathbb{R}) \Subset \langle \partial_t, \frac{4}{3}t\partial_t - x\partial_x, x^2u\partial_x, x^2\partial_x\rangle$.

- The realizations $sl^4(2,\mathbb{R})\oplus\langle\partial_t\rangle$, $sl^5(2,\mathbb{R})\oplus\langle\partial_t\rangle$ and $so^1(3)\oplus\langle\partial_t\rangle$ do not admit extensions to realizations admitted by equations of the form (1.1).

Summarizing the above results we give the final list of inequivalent equations from the class $\mathcal{C}_2$ that have nontrivial Levi factor:

$$sl^1(2,\mathbb{R}) \oplus \langle\partial_u\rangle : u_t = \frac{1}{4}u_x^2 + x^{-2}F(\omega),\quad \omega = x^2u_{xx} - xu_x;$$

$$sl^1(2,\mathbb{R}) \oplus \langle x\partial_x + 2u\partial_u\rangle : u_t = x^{-1}uu_x - x^{-2}u^2 + x^{-2}(2u - xu_x)^2F(\omega),\quad \omega = (x^2u_{xx} - 2u)(2u - xu_x)^{-1};$$

$$sl^2(2,\mathbb{R}) \oplus \langle\partial_u\rangle : u_t = -\frac{1}{4}x^{-1}u_x + x^{-3}u_x^{-1}F(\omega),\quad \omega = u_x^{-2}u_{xx} + 3x^{-1}u_x^{-1};$$

$$sl^3(2,\mathbb{R}) \oplus \langle\partial_t\rangle : u_t = xu_xF(\omega),\quad \omega = x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2};$$

$$sl^3(2,\mathbb{R}) \oplus \langle tx\partial_x\rangle : u_t = \frac{xu_x}{4t}\ln\left(x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2}\right) + xu_xF(t),$$

$$sl^4(2,\mathbb{R}) \oplus \langle\partial_t\rangle : u_t = x^{-2}\sqrt{4 + x^6u_x^2}\,F(\omega),\quad \omega = \left(4 + x^6u_x^2\right)^{-3/2}\left(x^4u_{xx} + 5x^3u_x + \frac{1}{2}x^9u_x^3\right);$$

$$sl^5(2,\mathbb{R}) \oplus \langle\partial_t\rangle : u_t = x^{-2}\sqrt{x^6u_x^2 - 4}\,F(\omega),\quad \omega = \left(x^6u_x^2 - 4\right)^{-3/2}\left(x^4u_{xx} + 5x^3u_x - \frac{1}{2}x^9u_x^3\right);$$

$$so^1(3) \oplus \langle\partial_t\rangle : u_t = \sqrt{\sec^2 x + u_x^2}\,F(\omega),\quad \omega = \left(1 + u_x^2\cos^2 x\right)^{-3/2}\left(u_{xx}\cos x - \left(2 + u_x^2\cos^2 x\right)u_x\sin x\right);$$

$$sl^1(2,\mathbb{R}) \oplus \langle\partial_u, x\partial_x + 2u\partial_u\rangle : u_t = \lambda x^2u_{xx}^2 - 2\lambda xu_xu_{xx} + \left(\lambda + \frac{1}{4}\right)u_x^2,\quad \lambda \neq 0;$$

$$sl^2(2,\mathbb{R}) \oplus \langle -u\partial_u, \partial_u\rangle : u_t = -\frac{1}{4}x^{-1}u_x + \lambda x^{-3}u_x^{-1}\left(u_x^{-2}u_{xx} + 3x^{-1}u_x^{-1}\right)^{-2},\quad \lambda \neq 0;$$

$$sl^3(2,\mathbb{R}) \oplus \langle -t\partial_t - mx\partial_x, \partial_t\rangle : u_t = \lambda xu_x\left|x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2}\right|^{1/4m},\quad \lambda \neq 0,\quad m \neq 0, \pm\frac{3}{4};$$

$$sl^3(2,\mathbb{R}) \oplus \langle t\partial_t, tx\partial_x\rangle : u_t = \frac{xu_x}{4t}\ln\left(x^{-5}u_x^{-3}u_{xx} + 2x^{-6}u_x^{-2}\right) + \frac{\lambda xu_x}{t},\quad \lambda \in \mathbb{R};$$

$$sl^1(2,\mathbb{R}) \Subset \langle t\partial_x + (tx^{-1} - x)\partial_u, \partial_x + x^{-1}u\partial_u\rangle :$$
$$u_t = \lambda u_{xx} + 2\lambda x^{-2}u - 2\lambda x^{-1}u_x + x^{-1}uu_x - x^{-2}u^2, \quad \lambda \neq 0;$$
$$sl^1(2,\mathbb{R}) \Subset \langle (tu + x^2)\partial_x + 2(xu + tx^{-1}u^2)\partial_u, u\partial_x + 2x^{-1}u^2\partial_u\rangle :$$
$$u_t = x^{-1}uu_x - x^{-2}u^2 + \lambda x^{-2}(2u - xu_x)(x^2u_{xx} + 2u - 2xu_x)^{-1}, \quad \lambda \neq 0;$$
$$sl^3(2,\mathbb{R}) \Subset \left\langle \partial_t, x\partial_x + \frac{4}{3}t\partial_t, x^{-1}\partial_u, -\partial_x + x^{-1}u\partial_u \right\rangle : u_t = x^{-1}(xu_{xx} + 2u_x)^{1/3};$$
$$sl^3(2,\mathbb{R}) \Subset \left\langle \partial_t, \frac{4}{3}t\partial_t - x\partial_x, x^2u\partial_x, x^2\partial_x \right\rangle : u_t = x^3u_x^2(xu_{xx} + 2u_x)^{-1/3}.$$

5. Concluding remarks

We present exhaustive description of invariant nonlinear evolution equations belonging to the class $\mathcal{C}_1$. We recall that the latter is formed by PDEs (1.1) that are not $\mathcal{E}$-equivalent to equations of the form

$$u_t = F(t, x, u_x, u_{xx}).$$

The corresponding inequivalent classes of invariant equations and their maximal invariance algebras are described by theorem 3.1. According to this theorem the list of inequivalent invariant equations from $\mathcal{C}_1$ consists of

- equation admitting one-dimensional Lie algebra;
- equation admitting two-dimensional solvable Lie algebra isomorphic to $A_{2.2}$;
- two equations admitting semi-simple Lie algebras isomorphic to $sl(2,\mathbb{R})$.

The invariant equations (1.1), which are equivalent to the above PDE, belong to the class $\mathcal{C}_2$. This division comes naturally if we take into account that equations from the class $\mathcal{C}_2$ can be transformed into quasi-linear evolution equations [21]. This fact can also be utilized to construct quasi-local symmetries of nonlinear evolution equations [31].

We give complete classification of PDEs from the class $\mathcal{C}_2$ invariant under the Lie algebras which are either semi-simple or semi-direct sums of semi-simple and solvable Lie algebras. In section 4, we provide the full list of invariant equations in question together with their maximal invariance algebras. It is comprised by equation (4.1), equations listed in lemmas 4.1, 4.2 and equations presented at the end of subsection 4.2. The algebraic properties of these equations can be summarized as follows. There are

- one equation admitting one-dimensional Lie algebra;
- four equations admitting semi-simple Lie algebras isomorphic to $sl(2,\mathbb{R})$ and $so(3)$;
- sixteen equations admitting symmetry algebras having nontrivial Levi factor.

To achieve a complete description of invariant equations of the form (1.1), we need to classify equations from $\mathcal{C}_2$ admitting solvable Lie algebras. This research is in progress now and will be reported in our future publications. In addition, we plan to use the results of group classification of equations from the class $\mathcal{C}_2$ in order to perform systematic study of quasi-local symmetries of PDEs (1.1) applying the approach developed in [31].

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